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The Theorem of Gleason for Nonseparable Hilbert Spaces

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Abstract

The Gleason theorem is proved for nonseparable Hilbert spaces under the assumption of the continuum hypothesis.

For an axiomatical foundation of quantum mechanics, see for instance Mackey (1963), Jauch (1968), Varadarajan (1968). The Gleason theorem is fundamental to characterize the states of a physical system. The theorem has been proven by Gleason (1957) in the case of a separable Hilbert space. In this paper we extend the proof to the nonseparable case. For that, we first consider an analogous theorem for states which are totally additive. (This has also been done by Guenin (footnote 1) and for the special case of pure states by Gudder (1972).) With the aid of the Ulam theorem (Ulam, 1930; Oxtoby, 1971) and assuming the continuum hypothesis we then show that every countably additive state is already totally additive. So, under the condition of the continuum hypothesis, the Gleason theorem is also valid for nonseparable Hilbert spaces.

Notations. Let H be an arbitrary Hilbert space. By $L(H)$ we denote the lattice of all closed subspaces of H. For the lattice operations we use the symbols $\wedge$ and $\vee$.

For $M \in L(H)$ let P^M denote the projection operator with range M.

A map $m:L(H) \to [0, 1]$ with $m(H) = 1$ is called a countably additive state (c state) if

$$m\left(\bigvee_{i=1}^{\infty} M_i\right) = \sum_{i=1}^{\infty} m(M_i) \tag{1}$$

[1] Private communication.

for every sequence $(M_i)_{i\in\mathbb{N}}$ of mutually orthogonal closed subspaces of H, and it is called a totally additive state (t state) if

$$m(\bigvee_{a\in I} M_a) = \sum_{a\in I} m(M_a) \tag{2}$$

for every family $(M_a)_{a\in I}$ with $M_a \perp M_b$ if $a \neq b$.

The trace class $T(H)$ of bounded operators of a nonseparable Hilbert space is defined in the same way as in the separable case, i.e., $W \in T(H)$ if for every orthonormal basis (onb) $\{\varphi_a | a \in I\}$ of H $\Sigma_{a\in I} \langle\varphi_a | W\varphi_a\rangle$ is absolutely convergent and independent of the basis used. For a positive operator W (this implies that there exists an operator A with $W = A^*A$)

$$tr\ W := \sum_{a\in I} \langle\varphi_a | W\varphi_a\rangle = \sum_{a\in I} \|A\varphi_a\|^2 \tag{3}$$

exists and is independent of the basis used if $\Sigma_{a\in I}\langle\varphi_a|W\varphi_a\rangle$ converges for some onb $\{\varphi_a|a\in I\}$ (cf. Schatten, 1970).

For $K \in L(H)$ we define

$$tr_K W := \sum_{b\in J} \langle\psi_b | W\psi_b\rangle \tag{4}$$

where $\{\psi_b | b \in J\}$ is an onb of K.

$W|_K$ denotes the restriction of W to K.

An operator $W: H \to H$ is called von Neumann operator if W is bounded, self-adjoint, nonnegative and of trace class. We say W is normed if $tr\ W = 1$.

Proposition 1. Let H be a Hilbert space and m a t state on $L(H)$. Then there exists a unique normed von Neumann operator W with

$$m(M) = tr\ P^M W \tag{5}$$

for all $M \in L(H)$.

Proof. Let $\{\varphi_a | a \in I\}$ be an onb of H.
Then

$$\sum_{a\in I} m(\text{span}\ \varphi_a) = m(H) = 1 \tag{6}$$

So there is a countable set $D \subset I$ with

$$m(\text{span}\ \varphi_a) = 0 \text{ for all } a \in I - D \tag{7}$$

We define

$$K_1 := \overline{\text{span}\ \{\varphi_a | a \in D\}} \tag{8}$$

K_1 is a separable Hilbert space and according to the Gleason theorem it follows that there is exactly one normed von Neumann operator $W_1' \in T(K_1)$ such that

$$m(M) = tr_{K_1} P^M W_1'\ \forall M \in L(K_1) \tag{9}$$

By the definition

$$W_1 := W_1' P^{K_1} \tag{10}$$

we get a normed von Neumann operator defined on the entire space H.

$$\left(tr\, W_1 = \sum_{a\in I} \langle \varphi_a | W_1' P^{K_1} \varphi_a \rangle = \sum_{a\in D} \langle \varphi_a | W_1' \varphi_a \rangle = 1\right) \tag{11}$$

It remains to be seen that W_1 also describes m correctly for closed subspaces M of H which are neither contained in K_1 nor in $K_1^\perp$.

Let M_2 be an arbitrary closed subspace of H. We can find an onb of M_2, which we complete to an onb of H, called $\{\psi_b | b \in J\}$

K_1 has countably many basis vectors φ_a. Each φ_a is represented by a countable linear combination $\Sigma_{i=1}^{\infty} c_i\psi_i$ of vectors ψ_i. So there is a countable set $C \subset J$ with

$$K_1 \subset \overline{\text{span}\, \{\psi_b | b \in C\}} =: K_2 \tag{12}$$

From $K_1 \subset K_2$ it follows that

$$m(K_2) = 1 \text{ and } m(K_2^\perp) = 0 \tag{13}$$

K_2 is a separable Hilbert space. So there is a unique normed von Neumann operator $W_2' \in T(K_2)$ with

$$m(M) = tr_{K_2} P^M W_2' \forall\, M \in L(K_2) \tag{14}$$

$$W_2 := W_2' P^{K_2}$$

A special case of (14) is

$$m(M) = tr_{K_2} P^M W_2 \forall\, M \in L(K_1)\, (\subset L(K_2)) \tag{15}$$

Clearly

$$tr_{K_2} P^M W_2 = tr_{K_1} P^M W_2 \forall\, M \in L(K_1) \tag{16}$$

On the other hand we have

$$m(M) = tr_{K_1} P^M W_1 \,\forall\, M \in L(K_1) \tag{17}$$

$W_1|_{K_1}$ and $W_2|_{K_1}$ are both normed von Neumann operators on K_1. On account of the uniqueness stated in the Gleason theorem we can conclude

$$W_1|_{K_1} = W_2|_{K_1} \tag{18}$$

$$tr_{K_1^\perp \wedge K_2} P^M W_2 = m(M) = 0 \quad \forall\, M \in L(K_1^\perp \wedge K_2) \tag{19}$$

Therefore

$$W_2|_{K_1^\perp \wedge K_2} = 0 \tag{20}$$

Since

$$W_2|_{K_2^\perp} = 0$$

we have

$$W_2|_{K_1^\perp} = 0 = W_1|_{K_1^\perp} \tag{21}$$

and together with (18)

$$W_1 = W_2 \tag{22}$$

From

$$M_2 = \overline{\text{span }\{\psi_b | b \in F\}} \qquad (\text{where } F \subset J) \tag{23}$$

it follows that

$$M_2 = \overline{\text{span }\{\psi_b | b \in F \cap C\}} \vee \overline{\text{span }\{\psi_b | b \in F \cap (J - C)\}} \tag{24}$$

hence

$$M_2 = (M_2 \wedge K_2) \vee (M_2 \wedge K_2^{\perp}) \tag{25}$$

This implies

$$\begin{aligned} m(M_2) &= m(M_2 \wedge K_2) + m(M_2 \wedge K_2^{\perp}) = \\ &= m(M_2 \wedge K_2) = \\ &= tr_H P^{M_2} W_2 = \\ &= tr_H P^{M_2} W_1 \end{aligned} \tag{26}$$

So we have shown

$$m(M) = tr\, P^M W_1 \quad \forall\ M \in L(H)$$

For the next theorem, we need some definitions (cf. Oxtoby, 1971).

A cardinal number A is called *weakly inaccessible* if

(a) A is greater than $\aleph_0$ ($\aleph_0$ denotes the cardinality of the set of all integers)

(b) A is not a successor (that means there is no cardinal B such that A is the smallest cardinal greater than B)

(c) if $B < A$, then A is not the sum of B cardinals each less than A.

A cardinal A is said to be *inaccessible* if A is weakly inaccessible and

(d) if $B < A$, then $\overline{\overline{P(B)}} < A$ (where $\overline{\overline{P(B)}}$ denotes the cardinality of the power set of B).

We call a cardinal number an *Ulam number* if it is not inaccessible.

The cardinals which can occur as dimensions of nonseparable Hilbert spaces in physics, c (the cardinality of the reals) and perhaps $2^c (2^{\overline{\overline{m}}} := \overline{\overline{P(m)}})$ are both Ulam numbers.

In the following, we will use the Ulam theorem. Let μ be a finite measure defined on all subsets of a set X with cardinality A, such that B is not weakly inaccessible for all cardinals $B \leqslant A$. Then μ vanishes identically if it is zero on every subset containing only one element.

One verifies easily that this is equivalent to the statement: Then there is a countable subset C of X with

$$\mu(X - C) = 0 \tag{27}$$

If one accepts the continuum hypothesis (saying $c = \aleph_1$) or only the weaker assumption: no cardinal $A \leqslant c$ is weakly inaccessible, then the Ulam theorem also holds for all sets the cardinality of which is an Ulam number.

Since the continuum hypothesis is independent of the other axioms of set theory (cf. Cohen, 1966) and it simplifies many set theoretical arguments, it is advantageous and harmless to accept it for physics and to add it to the axioms of set theory.

Proposition 2. Let H be a Hilbert space. If the cardinality of an onb (and this implies of every onb) of H is an Ulam number (especially c (or 2^c)) then every c state is totally additive.

Proof. Let $(M_a)_{a\in I}$ be a family of mutually orthogonal closed subspaces of H. We choose an onb in every M_a and complete their union to an onb $\{\psi_b | b \in J\}$ of H. It is easy to verify that by

$$\begin{aligned} \mu(A) &:= m(\overline{\text{span}\ \{\psi_b | b \in A\}}) \quad \text{if } A \neq \phi \\ \mu(\phi) &:= 0 \end{aligned} \tag{28}$$

a finite measure on $P(J)$ is defined. Employing the Ulam theorem we get a countable set $D \subset J$ with

$$\mu(J - D) = 0 \tag{29}$$

We define

$$\begin{aligned} J' &:= \{b \in J | \exists\, a \in I \text{ so that } \psi_b \in M_a\} \\ B &:= \{a \in I | M_a \wedge \overline{\text{span}\ \{\psi_b | b \in D\}} \neq \phi\} \end{aligned} \tag{30}$$

Then it follows that

$$\begin{aligned} m(\bigvee_{a\in I} M_a) &= m(\overline{\text{span}\ \{\psi_b | b \in J'\}}) = \\ &= \mu(J') = \\ &= \mu((J' \cap D) \cup (J' \cap (J - D))) = \\ &= \mu(J' \cap D) = \\ &= m(\overline{\text{span}\ \{\psi_b | b \in J' \cap D\}}) = \\ &= m(\bigvee_{a\in B} M_a) = \\ &= \sum_{a\in B} m(M_a) = \\ &= \sum_{a\in I} m(M_a) \end{aligned} \tag{31}$$

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References

Cohen, P. J. (1966). *Set Theory and the Continuum Hypothesis.* Benjamin, New York.
Gleason, A. M. (1957). *Journal of Mathematics and Mechanics,* **6**, 885–893.
Gudder, S. (1972). *International Journal of Theoretical Physics,* **6**, 369–375.
Jauch, J. (1968). *Foundations of Quantum Mechanics.* Addison-Wesley, Reading, Massachusetts.
Mackey, G. (1963). *Mathematical Foundations of Quantum Mechanics.* Benjamin, New York.
Oxtoby, J. C. (1971). *Maß und Kategorie,* pp. 29–31. Springer, Berlin–Heidelberg–New York.
Schatten, R. (1970). *Norm Ideals of Completely Continuous Operators,* p. 29, lemma 1. Springer, Berlin–Heidelberg–New York.
Ulam, S. (1930). *Fundamenta Mathematicae,* **16**, 140-150.
Varadarajan, V. (1968). *Geometry of Quantum Theory,* Vol. I, Van Nostrand, Princeton.